Relations between Metric Dimension and Domination Number of Graphs

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Abstract

A set $W \subseteq V(G)$ is called a resolving set, if for each two distinct vertices $u, v \in V(G)$ there exists $w \in W$ such that $d(u, w) \neq d(v, w)$, where $d(x, y)$ is the distance between the vertices x and y. The minimum cardinality of a resolving set for G is called the metric dimension of G, and denoted by $\beta(G)$. In this paper, we prove that in a connected graph G of order n, $\beta(G) \leq n - \gamma(G)$, where $\gamma(G)$ is the domination number of G, and the equality holds if and only if G is a complete graph or a complete bipartite graph $K_{s,t}$, $s, t \geq 2$. Then, we obtain new bounds for $\beta(G)$ in terms of minimum and maximum degree of G.

Keywords: Resolving set; Metric dimension; Dominating set; Domination number.

1 Introduction

Throughout the paper, $G = (V, E)$ is a finite, simple, and connected graph of order n. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. The diameter of G, denoted by $diam(G)$ is $\max\{d(u, v) \mid u, v \in V\}$. The set of all neighbors of a vertex v is denoted by $N(v)$. The maximum degree and minimum degree of graph G, are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and non-adjacency relations between u and v, respectively.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the *metric representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices have different representations. A resolving set for G with minimum cardinality is called a *metric basis*, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$.

It is obvious that to see whether a given set W is a resolving set, it is sufficient to consider the vertices in $V(G)\backslash W$, because $w \in W$ is the unique vertex of G for which $d(w, w) = 0$. When W

is a resolving set for G, we say that W *resolves* G. In general, we say an ordered set W resolves a set $T \subseteq V(G)$ of vertices in G, if for each two distinct vertices $u, v \in T$, $r(u|W) \neq r(v|W)$.

In [12], Slater introduced the idea of a resolving set and used a *locating set* and the *location number* for what we call a resolving set and the metric dimension, respectively. He described the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [6] discovered the concept of the location number as well and called it the metric dimension. For more results related to these concepts see [2, 3, 5, 8, **?**]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [11], network discovery and verification [1], robot navigation [8], mastermind game [2], problems of pattern recognition and image processing [10], and combinatorial search and optimization [11].

The following bound is the known upper bound for metric dimension.

Theorem A. [4] *If G is a connected graph of order n, then $\beta(G) \leq n - diam(G)$.*

A set $\Gamma \subseteq V(G)$ is a *dominating set* for G if every vertex not in Γ has a neighbor in Γ. A dominating set with minimum size is a *minimum dominating set* for G. The *domination number* of G, $\gamma(G)$, is the cardinality of a minimum dominating set. In Section 2, we prove that $\beta(G) \leq n - \gamma(G)$. Moreover, we prove that $\beta(G) = n - \gamma(G)$ if and only if G is a complete graph or a complete bipartite graph $K_{s,t}$, $s, t \geq 2$. In Section 3, regarding to known bounds of $\gamma(G)$, we obtain new upper bounds for metric dimension in terms of other graph parameters.

2 Main Results

In this section, we prove that $\beta(G) \leq n - \gamma(G)$. Moreover, we show that $\beta(G) = n - \gamma(G)$ if and only if G is a complete graph or a complete bipartite graph $K_{s,t}$, $s, t \geq 2$.

Two vertices $u, v \in V(G)$ are called *false twin* vertices if $N(u) = N(v)$.

Lemma 1. *Let G be a connected graph. Then there exists a minimum dominating set for G which does not have any pair of false twin vertices.*

Proof. Let Γ be a minimum dominating set for G with minimum number of false twin pairs of vertices and u, v be an arbitrary false twin pair in Γ. Since u and v dominate the same vertices in G, they have no neighbors in Γ; otherwise, $\Gamma \setminus \{u\}$ and $\Gamma \setminus \{v\}$ are dominating sets in G which is a contradiction. On the hand, G is connected, hence u and v have some neighbors in $V(G) \setminus \Gamma$. Now, $\Gamma' = \Gamma \cup \{x\} \setminus \{u\}$, where x is a neighbor of u in $V(G) \setminus \Gamma$, is a dominating set for G with fewer number of false twin pair of vertices. This contradiction implies that Γ has no false twin pair of vertices. ■

Theorem 1. *For every connected graph G of order n, $\beta(G) \leq n - \gamma(G)$. In particular, if Γ is a minimum dominating set for G with no false twin pair of vertices, then $V(G) \setminus \Gamma$ is a resolving set for G.*

Proof. By Lemma 1, G has a minimum dominating set Γ with no pair of false twin vertices. Suppose, on the contrary, that $V(G) \setminus \Gamma$ is not a resolving set for G. Then, there exist vertices u and v in Γ such that $r(u|V(G) \setminus \Gamma) = r(v|V(G) \setminus \Gamma)$. This implies that all neighbors of u and v in $V(G) \setminus \Gamma$ are the same. Therefore, u and v have no neighbor in Γ; otherwise we can remove one of the vertices u and v from Γ and get a dominating set with cardinality $|\Gamma| - 1$. Hence, u and v are false twin vertices, which is a contradiction. Thus, $V(G) \setminus \Gamma$ is a resolving set for G. Accordingly, $\beta(G) \leq n - \gamma(G)$. ∎

The following example shows that Theorem 1 gives a better upper bound for $\beta(G)$ comparing the upper bound in Theorem A.

Example 1. *Let G be a connected graph of order $3k + 1$, $k \geq 6$, obtained from the wheel W_k by replacing each spoke by a path of length three. It is easy to see that $\gamma(G) = k+1$, by Theorem 1, $\beta(G) \leq n - \gamma(G) = 2k$ while $diam(G) \leq 6$ and by Theorem A, $\beta(G) \leq 3k + 1 - diam(G)$.*

In the sequel we need the following definition.

Definition 1. *Let Γ be a dominating set in a connected graph G and $u \in \Gamma$. A vertex $\bar{u} \in V(G) \setminus \Gamma$ is called a private neighbor of u if u is the unique neighbor of $\bar{u}$ in Γ, i.e., $N(\bar{u}) \cap \Gamma = \{u\}$.*

It is clear that each vertex of a minimum dominating set Γ for a graph G has a private neighbor or it is a single vertex in Γ. The following lemma provides a minimum dominating set Γ for G with no false twin pair of vertices such that every vertex in Γ has a private neighbor.

Lemma 2. *Every connected graph G has a minimum dominating set Γ with no false twin pair of vertices such that every vertex in Γ has a private neighbor.*

Proof. By Lemma 1, let Γ be a minimum dominating set with no false twin pair of vertices with minimum number of single vertices. Also, let u be a single vertex in Γ. Since G is a connected graph, u has a neighbor in $V(G) \setminus \Gamma$, say x. Now $\Gamma' = \Gamma \cup \{x\} \setminus \{u\}$ is also a minimum dominating set for G with no false twin pair of vertices, because x is the unique vertex in Γ' that is adjacent to u. Moreover, u is a private neighbor of x in $V(G) \setminus \Gamma'$. Note that, x was not a private neighbor of any vertex in Γ. Therefore, the number of vertices in Γ' which have a private neighbor is more than the number of vertices in Γ which have a private neighbor in $V(G) \setminus \Gamma$. On the other words, the number of single vertex in Γ' is fewer than Γ. This contradiction implies that all vertices in Γ have a private neighbor in $V(G) \setminus \Gamma$. ∎

Theorem 2. *Let G be a connected graph of order n. Then $\beta(G) = n - \gamma(G)$ if and only if $G = K_n$ or $G = K_{s,t}$, for some $s, t \geq 2$.*

Proof. Clearly, for $G = K_n$ and $G = K_{s,t}$, $s, t \geq 2$, the equality holds. Now let $\beta(G) = n - \gamma(G)$. By Lemma 2, there exists a minimum dominating set Γ for G with no false twin vertices such that all vertices in Γ have a private neighbor in $V(G) \setminus \Gamma$. Let $\Gamma = \{u_1, u_2, \ldots, u_r\}$ and $W_1 = \{x_1, x_2, \ldots, x_r\}$, where x_i is private neighbor of u_i for an i, $1 \leq i \leq r$. Since u_i is the

unique neighbor of x_i in Γ, for each i, j, $1 \le i, j \le r$, the i^{th} coordinate of $r(u_j|W_1)$ is 1 if and only if $j = i$. Therefore, W_1 resolves the set Γ.

By Theorem 1, $W = V(G) \setminus \Gamma$ is a resolving set for G and $\beta(G) = n - \gamma(G)$ implies that W is a metric basis. Now let $x \in W \setminus W_1$. Since W_1 resolves Γ, there exists a unique vertex $u_i \in \Gamma$ such that $r(x|W_1) = r(u_i|W_1)$. Thus, x and u_i have the same neighbors in W_1, but $N(u_i) \cap W_1 = \{x_i\}$, hence $N(x) \cap W_1 = \{x_i\}$. Thus, $W \setminus W_1$ is partitioned into sets $V_1, V_2, \ldots, V_r$, (some V_i's could be empty) such that for each i, $1 \le i \le r$, and every $x \in V_i$, $N(x) \cap W_1 = \{x_i\}$. Therefore, W_1 is a minimum dominating set for G. Moreover, W_1 has no pair of false twin vertices, because for each i, $1 \le i \le r$, x_i is the unique neighbor of u_i in W_1. Hence, by Theorem 1 the set $B = V(G) \setminus W_1$ is a metric basis of G.

Now let $B_i = V_i \cup \{u_i\}$. For a fixed i, $1 \le i \le r$, let a be an arbitrary vertex in B_i. Since B is a metric basis of G, $B \setminus \{a\}$ is not a resolving set for G. Therefore, there exists a vertex $x_{j_a} \in W_1$ such that $r(a|B \setminus \{a\}) = r(x_{j_a}|B \setminus \{a\})$. If $j_a = i$, then a is adjacent to all vertices in $B_i \setminus \{a\}$, since x_i is adjacent to all vertices in B_i. If $j_a \ne i$, then a is not adjacent to any vertex in B_i, since x_j, $j \ne i$, is not adjacent to any vertex in B_i. Hence, for every two vertices a and a' in B_i, where $j_a = j_{a'}$. Thus, we conclude that, for every vertex $a \in B_i$, there exists a vertex $x_j \in W_1$ such that $r(a|B \setminus \{a\}) = r(x_j|B \setminus \{a\})$, and there are two possibilities $j = i$ or $j \ne i$; in the former case B_i is a clique and in the latter case B_i is an independent set.

Now let there exists i, $1 \le i \le r$, such that for every vertex $a \in B_i$, $r(a|B \setminus \{a\}) = r(x_i|B \setminus \{a\})$. It was shown that in this case B_i is a clique. Moreover, since a is not adjacent to any vertex in $W_1 \setminus \{x_i\}$, x_i is not adjacent to any vertex in $W_1 \setminus \{x_i\}$. Moreover, since x_i is not adjacent to any vertex in $B \setminus B_i$, a is not adjacent to any vertex in $B \setminus B_i$. Therefore, the induced subgraph by $B_i \cup \{x_i\}$ is a maximal connected subgraph of G. Since G is a connected graph, $G = B_i \cup \{x_i\}$, and consequently $G = K_n$.

Otherwise, for each i, $1 \le i \le r$, and for every vertex $a \in B_i$, $r(a|B \setminus \{a\}) \ne r(x_i|B \setminus \{a\})$ and $r(a|B \setminus \{a\}) = r(x_j|B \setminus \{a\})$ for some $j \ne i$. Now, for each $b \in B_j$, if $r(b|B \setminus \{b\}) = r(x_k|B \setminus \{b\})$, then x_k is adjacent to all vertices in B_i, since b is adjacent to all vertices in B_i. Thus, $k = i$. It was shown that in this case each B_i, $1 \le i \le r$, is an independent set. Now, since x_j is adjacent to all vertices in B_j, every vertex $a \in B_i$ is adjacent to all vertices in B_j. Therefore, the induced subgraph $B_i \cup B_j$ is a complete bipartite graph.

Note that, each vertex in $B_i \cup B_j$ is not adjacent to any vertex in B_k, $k \in \{1, 2, \ldots, r\} \setminus \{i, j\}$, because x_i and x_j are not adjacent to any vertex in $B \setminus (B_i \cup B_j)$. On the other hand, x_i and x_j are not adjacent to any vertex in $W_1 \setminus \{x_i, x_j\}$, since all vertices in $B_i \cup B_j$ are not adjacent to any vertex in this set. Therefore, the induced subgraph by $B_i \cup B_j \cup \{x_i, x_j\}$ is a maximal connected subgraph of G. Since G is a connected graph, $G = B_i \cup B_j \cup \{x_i, x_j\}$. Furthermore, $r(a|B \setminus \{a\}) = r(x_j|B \setminus \{a\})$ implies that $x_i \sim x_j$, because $a \sim x_i$. Thus, $G = K_{s,t}$. Since $u_i \in B_i$ and $u_j \in B_j$, $s, t \ge 2$. ■

3 Upper Bounds for the Metric Dimension

The domination number is a well studied parameter and there are several bounds for $\gamma(G)$ in terms of the other graph parameters. Following the given new upper bound for $\beta(G)$ in

Theorem 1, several new upper bounds for metric dimension can be obtained. In what follows, we present some of these new upper bounds.

Theorem B. [7] *For every graph G of order n and girth g,*

(i) *if $g \geq 5$, then $\gamma(G) \geq \delta(G)$.*

(ii) *if $g \geq 6$, then $\gamma(G) \geq 2(\delta(G) - 1)$.*

(iii) $\gamma(G) \geq \left\lceil \frac{n}{1+\Delta(G)} \right\rceil$.

(iv) *If G has degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_i \geq d_{i+1}$, then $\gamma(G) \geq \min\{k \mid k + (d_1 + d_2 + \cdots + d_k) \geq n\}$.*

(v) *if $\delta(G) \geq 2$ and $g \geq 7$, then $\gamma(G) \geq \Delta(G)$.*

Theorem C. [9] *Let $\mu_n \geq \mu_{n-1} \geq \cdots \geq \mu_1$ be the eigenvalues of Laplacian matrix of connected graph G of order $n \geq 2$, then $\gamma(G) \geq \frac{n}{\mu_n(G)}$.*

By Theorem 1 and above theorems, the list of new upper bounds for metric dimension in terms of other graph parameters are obtained.

Corollary 1. *For every connected graph G of order n and girth g,*

(i) *if $g \geq 5$, then $\beta(G) \leq n - \delta(G)$.*

(ii) *if $g \geq 6$, then $\beta(G) \leq n - 2\delta(G) + 2$.*

(iii) $\beta(G) \leq n(G) - \left\lceil \frac{n}{1+\Delta(G)} \right\rceil$.

(iv) *if G has degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_i \geq d_{i+1}$, then $\beta(G) \leq n - \min\{k \mid k + (d_1 + d_2 + \cdots + d_k) \geq n\}$.*

(v) *if $\mu_n \geq \mu_{n-1} \geq \cdots \geq \mu_1$ be the eigenvalues of Laplacian matrix of G, then $\beta(G) \leq n - \frac{n}{\mu_n(G)}$.*

(vi) *if $\delta(G) \geq 2$ and $g \geq 7$, then $\beta(G) \leq n - \Delta(G)$.*

For each of the given upper bounds in above, infinite classes of graphs can be constructed to show that these bounds could be better than $n - diam(G)$.

In the following example, we consider the well known graph Kneser $KG(2k+1, k)$, which is called odd graph. The Kneser graph with integer parameters n and k, $n \geq 2k$, denoted by $KG(n, k)$, is the graph with k element subsets of set $\{1, 2, \ldots, n\}$ as the vertex set and two vertices are adjacent if and only if the corresponding subsets are disjoint.

Example 2. *Let $G = KG(2k+1, k)$, for $k \geq 3$. Then, $n = |V(G)| = \binom{2k+1}{k}$, $\Delta(G) = \delta(G) = k+1$, $g(G) = 6$, $\mu_{\binom{2k+1}{k}}(G) = 2k+1$, and $diam(G) = k$. Therefore, we have:*

(i) $\beta(G) \leq n - k - 1$.

(ii) $\beta(G) \leq n - 2k$.

(iii) $\beta(G) \leq n - \left\lceil \frac{\binom{2k+1}{k}}{k+2} \right\rceil$.

(iv) $\beta(G) \leq n - \frac{\binom{2k+1}{k}}{k+2}$.

(v) $\beta(G) \leq n - \frac{\binom{2k+1}{k}}{2k+1}$.

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